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The fields of a moving point charge: a new derivation from Jefimenko's equations

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Abstract
We give a derivation for the fields of a moving point charge from Jefimenko's equations without the usual cumbersome differentiation of retarded quantities.

1. Introduction

A discussion of electric and magnetic fields generated by an arbitrarily moving point charge is part of all textbooks on electrodynamics. The usual approaches are either to calculate them from the Lienard–Wiechert potentials or to use the more elegant approach invoking special relativity [1]. The first, traditional, path is more complicated but, at least for the case of a charge moving at constant velocity, can be found in all textbooks. In addition, it provides a satisfying sense of closure to the nonrelativistic domain of a classical theory which is not complete without generalization to arbitrary velocity. In the most direct, 'brute force' form [2], this method is seldom presented in full detail for an arbitrarily moving charge even in advanced electrodynamic textbooks because it involves first computing the vector and scalar potentials and then evaluating the field strengths by taking appropriate derivatives. This forces one to compute implicitly the derivatives of the retarded time with respect to the position and time variables; an operation which is extremely cumbersome (some authors even have said this is a 'heroic' calculation). According to Jefimenko [3] it 'constitutes one of the most complicated procedures in classical electrodynamic theory'. It is very time consuming and it is doubtful that such a derivation has actually ever been fully discussed during lectures.

Recently there has been renewed interest in this problem, partially, because of its *apparent* calculational complexity. Many authors [4–7] had considered this non-relativistic approach as a kind of a test problem to demonstrate the advantage of using Jefimenko's equations [8] for direct calculation of electric **E** and magnetic **B** fields

$$\mathbf{E}(\mathbf{r}, t) = \int \mathrm{d}\mathbf{r}' \left(\frac{[\rho(\mathbf{r}', t')]\hat{\mathbf{R}}}{R^2} + \frac{\frac{\partial}{\partial t}[\rho(\mathbf{r}', t')]\hat{\mathbf{R}}}{cR} - \frac{\frac{\partial}{\partial t}[\mathbf{j}(\mathbf{r}', t')]}{c^2 R} \right), \tag{1}$$

$$\mathbf{B}(\mathbf{r}, t) = \int \mathrm{d}\mathbf{r}' \left(\frac{[\mathbf{j}(\mathbf{r}', t')] \times \hat{\mathbf{R}}}{cR^2} + \frac{\frac{\partial}{\partial t}[\mathbf{j}(\mathbf{r}', t')] \times \hat{\mathbf{R}}}{c^2 R} \right), \tag{2}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $R = |\mathbf{R}|$, $\hat{\mathbf{R}} = \mathbf{R}/R$ and square brackets denote retarded value which is a short form for t' integration of source function with δ-function $\left(\delta\left(t - t' - \frac{R}{c}\right)\right)$. The computation sometimes is easier if the explicit integral form is retained, as is discussed below.

The advantages of using Jefimenko's equations were discussed in [4, 5, 9]. They are quite obvious and are apparent from equations (1) and (2). One can skip the two stage calculations involving potentials so that spatial derivatives (the longest part in direct calculations with potentials [2]) are not present. This is a great simplification, especially for calculations with retarded quantities. Unfortunately, these initial advantages are partially lost in subsequent calculations of the fields in [4–6, 10, 11] because the direct procedure (as a direct procedure for potentials) was employed. A more economical method was used previously in calculations with potentials in [12–14] and, actually, in the derivation of Jefimenko's equations in [9].

The main idea of this latter approach is, in general, to work with integrals over space and time to avoid the problem of operating on retarded quantities and, in particular, for point charges, to perform spatial integrations first. We will demonstrate how this idea can simplify calculations, but before doing so we would like to make the following comment.

In standard courses we (instructors) are teaching material (results) that is established and well known. In contrast to research, where a nice derivation is good to have but the interesting result is much more important, the teaching is not only about the result (what to teach) but also about its delivery (how to teach), including the search for rigorous, economical and transparent derivations. From this point of view, the discussion of even a very small improvement in delivery of standard material is much more important than some complicated calculations that will be never seen by students. The goal of this paper is to provide *an improved derivation* for the fields of a moving particle (point charge) from Jefimenko's equations which is short, rigorous, transparent and can be taught in less than one lecture. In the next section we will present the derivation and compare, in more detail, our approach with previous calculations.

2. Derivation

For a point charge e, the charge ρ and current $\mathbf{j}$ densities are given in terms of the δ-function by

$$\rho(\mathbf{r}, t) = e\delta(\mathbf{r} - \mathbf{s}(t)), \tag{3}$$

$$\mathbf{j}(\mathbf{r}, t) = e\frac{\mathrm{d}\mathbf{s}(t)}{\mathrm{d}t}\delta(\mathbf{r} - \mathbf{s}(t)) = \mathbf{v}(t)\rho(\mathbf{r}, t), \tag{4}$$

where $\mathbf{s}(t)$ is the position of the point charge.

The first rather detailed derivation of the fields of a point charge from Jefimenko's equations was presented by Griffiths and Heald [4]. (It is also given in [6].) They started from equations (1), (2) (i.e., using the retarded charge and current densities), which is not the best approach because it is necessary to work with retarded quantities and perform integration with the δ-function of a spatial argument, in which case the change of a vector variable involves quite a long calculation of a Jacobian [15]. A better approach is to make the implicit dependence explicit by evaluating time integral after performing the spatial integration [9, 13, 14].

We first consider the derivation of the electric field $\mathbf{E}$ of a moving point particle by using equation (1). Upon substitution of equations (3), (4) we obtain

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) = e\int \mathrm{d}\mathbf{r}'\int \mathrm{d}t'\frac{\delta(\mathbf{r}'-\mathbf{s}(t'))\hat{\mathbf{R}}}{R^2}\delta\left(t-t'-\frac{R}{c}\right)\\ +e\frac{\partial}{\partial t}\int \mathrm{d}\mathbf{r}'\int \mathrm{d}t'\frac{\delta(\mathbf{r}'-\mathbf{s}(t'))\hat{\mathbf{R}}}{cR}\delta\left(t-t'-\frac{R}{c}\right)\\ -e\frac{\partial}{\partial t}\int \mathrm{d}\mathbf{r}'\int \mathrm{d}t'\frac{\mathbf{v}(t')\delta(\mathbf{r}'-\mathbf{s}(t'))}{c^2R}\delta\left(t-t'-\frac{R}{c}\right).\end{aligned} \tag{5}$$

Performing the spatial integration, we obtain

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) = e\int \mathrm{d}t'\frac{\hat{\mathbf{R}}_s}{R_s^2}\delta\left(t-t'-\frac{R_s}{c}\right)+e\frac{\partial}{\partial t}\int \mathrm{d}t'\frac{\hat{\mathbf{R}}_s}{cR_s}\delta\left(t-t'-\frac{R_s}{c}\right)\\ -e\frac{\partial}{\partial t}\int \mathrm{d}t'\frac{\mathbf{v}(t')}{c^2R_s}\delta\left(t-t'-\frac{R_s}{c}\right).\end{aligned} \tag{6}$$

The $\mathbf{R}$ in (5) became $\mathbf{R}_s = \mathbf{r}-\mathbf{s}(t')$ in (6). The idea behind this method is to perform the retardation integral at a late stage to postpone the calculations with retarded quantities. The integration in (6) is easily performed by the change of a single variable [16, 17] or just using the well-known formula

$$\delta(f(t')) = \sum_i \frac{\delta(t-t_i)}{\left|\frac{\mathrm{d}f}{\mathrm{d}t'}\right|_{t'=t_i}}, \tag{7}$$

where t_i are the zeros of $f(t')$. Applying equation (7) to our case yields

$$\left|\frac{\mathrm{d}f}{\mathrm{d}t'}\right| = \left(1+\frac{1}{c}R_s'\right) = K = \left(1-\frac{1}{c}\mathbf{v}\cdot\hat{\mathbf{R}}_s\right), \tag{8}$$

where the prime on R_s indicates differentiation with respect to t'. Finally, we obtain

$$\mathbf{E}(\mathbf{r},t) = e\left[\frac{\hat{\mathbf{R}}_s}{KR_s^2}\right]+\frac{e}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}_s}{KR_s}\right]-\frac{e}{c^2}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}(t')}{KR_s}\right]. \tag{9}$$

Equation (9) is well known, but it is not a final result and still needs the implicit differentiation to obtain the Lienard–Wiechert fields.

Is this as far as we can go or can we further simplify calculations by postponing the operations with implicit differentiation? The answer is 'yes'. Let us return to equation (6). Putting the t-differentiation inside the integral, remembering that only the δ-function has t in its argument, and using the following property, which is not difficult to prove by using Fourier representation of δ-function

$$\begin{aligned}\frac{\partial}{\partial t'}\delta\left(t-t'-\frac{R_s}{c}\right) &= \frac{\partial}{\partial t'}\int \mathrm{d}\omega\exp\left\{\mathrm{i}\omega\left(t-t'-\frac{R_s}{c}\right)\right\}\\ &= \int \mathrm{d}\omega\,\mathrm{i}\omega\frac{\partial}{\partial t'}\left(t-t'-\frac{R_s}{c}\right)\exp\left\{\mathrm{i}\omega\left(t-t'-\frac{R_s}{c}\right)\right\}\\ &= -\left(1+\frac{1}{c}R_s'\right)\int \mathrm{d}\omega\,\mathrm{i}\omega\exp\mathrm{i}\omega\left(t-t'-\frac{R_s}{c}\right)\\ &= -\left(1+\frac{1}{c}R_s'\right)\frac{\partial}{\partial t}\delta\left(t-t'-\frac{R_s}{c}\right),\end{aligned}$$

we have

$$\frac{\partial}{\partial t}\delta\left(t - t' - \frac{R_s}{c}\right) = -\frac{1}{\left(1 + \frac{1}{c}R'_s\right)}\frac{\partial}{\partial t'}\delta\left(t - t' - \frac{R_s}{c}\right). \tag{10}$$

Making the substitution of equation (10) into (6) and carrying out the integration by parts in the last two terms we obtain

$$\mathbf{E}(\mathbf{r}, t) = e\int \mathrm{d}t' \left(\frac{\hat{\mathbf{R}}_s}{R_s^2} + \frac{\partial}{\partial t'}\left(\frac{1}{\left(1 + \frac{1}{c}R'_s\right)}\left(\frac{\hat{\mathbf{R}}_s}{cR_s} - \frac{\mathbf{v}(t')}{c^2R_s}\right)\right)\right)\delta\left(t - t' - \frac{R_s}{c}\right), \tag{11}$$

where all differentiations are still explicit! Only after performing all operations we, at the final stage, integrate with respect to t'

$$\mathbf{E}(\mathbf{r}, t) = e\left[\frac{1}{\left(1 + \frac{1}{c}R'_s\right)}\left(\frac{\hat{\mathbf{R}}_s}{R_s^2} + \frac{\partial}{\partial t'}\left(\frac{1}{\left(1 + \frac{1}{c}R'_s\right)}\left(\frac{\hat{\mathbf{R}}_s}{cR_s} - \frac{\mathbf{v}(t')}{c^2R_s}\right)\right)\right)\right]. \tag{12}$$

In obtaining equation (12) *all* implicit differentiations have been avoided making the derivation simple, short and transparent.

Similarly, from equation (2) we obtain

$$\mathbf{B}(\mathbf{r}, t) = e\left[\frac{1}{\left(1 + \frac{1}{c}R'_s\right)}\left(\frac{\mathbf{v}(t') \times \hat{\mathbf{R}}_s}{cR_s^2} + \frac{\partial}{\partial t'}\left(\frac{1}{\left(1 + \frac{1}{c}R'_s\right)}\frac{\mathbf{v}(t') \times \hat{\mathbf{R}}_s}{c^2R_s}\right)\right)\right]. \tag{13}$$

Let us calculate only the contribution to the 'acceleration' electric field $\mathbf{E}_a$ from (12) (i.e. keeping only terms proportional to $\dot{\mathbf{V}}$, where $\mathbf{V}(t') = \mathbf{v}(t')/c$) to demonstrate the simplicity and power of this technique

$$\begin{aligned}\mathbf{E}_a(\mathbf{r}, t) &= \frac{e}{c}\left[\frac{\hat{\mathbf{R}}_s - \mathbf{V}}{(1 - \mathbf{V}\cdot\hat{\mathbf{R}}_s)R_s}\frac{\partial}{\partial t'}\frac{1}{(1 - \mathbf{V}\cdot\hat{\mathbf{R}}_s)} - \frac{\dot{\mathbf{V}}}{(1 - \mathbf{V}\cdot\hat{\mathbf{R}}_s)^2R_s}\right] \\ &= \frac{e}{c}\left[\frac{1}{(1 - \mathbf{V}\cdot\hat{\mathbf{R}}_s)^3R_s}((\hat{\mathbf{R}}_s - \mathbf{V})\dot{\mathbf{V}}\cdot\hat{\mathbf{R}}_s - \dot{\mathbf{V}}(1 - \mathbf{V}\cdot\hat{\mathbf{R}}_s))\right],\end{aligned} \tag{14}$$

where, with $1 = \hat{\mathbf{R}}_s \cdot \hat{\mathbf{R}}_s$, we have

$$(\hat{\mathbf{R}}_s - \mathbf{V})\dot{\mathbf{V}}\cdot\hat{\mathbf{R}}_s - \dot{\mathbf{V}}((\hat{\mathbf{R}}_s - \mathbf{V})\cdot\hat{\mathbf{R}}_s) = \hat{\mathbf{R}}_s \times \{(\hat{\mathbf{R}}_s - \mathbf{V}) \times \dot{\mathbf{V}}\}.$$

This is exactly the second term in the well-known expression [1] (equation (14.14)):

$$\mathbf{E}(\mathbf{r}, t) = e\left[\frac{(\hat{\mathbf{R}}_s - \mathbf{V})(1 - V^2)}{(1 - \mathbf{V}\cdot\hat{\mathbf{R}}_s)^3R_s^2}\right] + \frac{e}{c}\left[\frac{\hat{\mathbf{R}}_s \times \{(\hat{\mathbf{R}}_s - \mathbf{V}) \times \dot{\mathbf{V}}\}}{(1 - \mathbf{V}\cdot\hat{\mathbf{R}}_s)^3R_s}\right]. \tag{15}$$

Similarly from equation (13) we obtain ([6], equations (7.37c) and (7.37d))

$$\mathbf{B}(\mathbf{r}, t) = e\left[\frac{(\mathbf{V} \times \hat{\mathbf{R}}_s)(1 - V^2)}{(1 - \mathbf{V}\cdot\hat{\mathbf{R}}_s)^3R_s^2}\right] + \frac{e}{c}\left[\frac{(\dot{\mathbf{V}}\cdot\hat{\mathbf{R}}_s)(\mathbf{V} \times \hat{\mathbf{R}}_s) + (\dot{\mathbf{V}} \times \hat{\mathbf{R}}_s)(1 - \mathbf{V}\cdot\hat{\mathbf{R}}_s)}{(1 - \mathbf{V}\cdot\hat{\mathbf{R}}_s)^3R_s}\right]. \tag{16}$$

Thus the magnetic field is always perpendicular to **E** and retarded radius vector

$$\mathbf{B}(\mathbf{r},t) = [\hat{\mathbf{R}}_s] \times \mathbf{E}(\mathbf{r},t). \tag{17}$$

This relation is obvious for 'velocity' parts of the fields (compare equations (15), (16)) and for 'acceleration' parts it is easy to see from intermediate result (14).

Note that in the literature two slightly different (but, of course, equivalent) forms exist: $\frac{\partial}{\partial t}[..]$ or $\left[\frac{\partial \ldots}{\partial t'}\right]$. The last form is due to Panofsky and Phillips [18] (see also [19]). We used the first one to demonstrate the connection to traditional derivations, but the second one can also be used. In addition, it is instructive to do this in another way to demonstrate some properties of δ-functions. For example, let us consider the last term in equation (5) written in the second form

$$-e\int \mathrm{d}\mathbf{r}' \int \mathrm{d}t' \frac{\partial}{\partial t'}\{\mathbf{v}(t')\delta(\mathbf{r}'-\mathbf{s}(t'))\}\frac{1}{c^2R}\delta\left(t-t'-\frac{R}{c}\right).$$

Integration by parts with respect to t' gives

$$+e\int \mathrm{d}\mathbf{r}' \int \mathrm{d}t' \{\mathbf{v}(t')\delta(\mathbf{r}'-\mathbf{s}(t'))\}\frac{1}{c^2R}\frac{\partial}{\partial t'}\delta\left(t-t'-\frac{R}{c}\right).$$

It is incorrect to perform spatial integration in this equation (such would give the wrong answer) because of the simultaneous presence of time dependence in the argument of the spatial δ-function and derivative of the second one. We have to eliminate this derivative before performing the spatial integration,

$$\frac{\partial}{\partial t}\delta\left(t-t'-\frac{R}{c}\right) = -\frac{\partial}{\partial t'}\delta\left(t-t'-\frac{R}{c}\right).$$

Now, performing spatial integration, we have

$$-e\int \mathrm{d}t'\mathbf{v}(t')\frac{1}{c^2R_s}\frac{\partial}{\partial t}\delta\left(t-t'-\frac{R_s}{c}\right).$$

After using equation (10) and finally integrating by parts we immediately recover equation (12) as we should. This also demonstrates that this approach can even compete with the relativistic one in terms of simplicity and clarity.

From the literature concerning the application of Jefimenko's equations for calculation of fields of a moving charge particle, the result of Bellotti and Bornatici deserves special attention [7]. The major focus of their article was the alternative derivation of Jefimenko's equations based on a Fourier transform method and, as an example of its application, the fields of an arbitrarily moving point charge were also considered. To calculate the fields they also did not use equations (1), (2) (the standard form) and went back to $(\mathbf{r}, \omega)$-space. The proof provided is a little bit sketchy and not very transparent but our equations (12), (13) are equivalent to equations (17), (18) of their article. From the citation search, it looks like their result was not recognized. Even authors of articles that refer to this work consider it as an alternative derivation of Jefimenko's equations only (we have to confess that we also recognized their result only after performing our derivation) and continue to perform implicit differentiation in calculating the fields of moving dipoles [20], which is extremely time consuming. This is especially true for fields of a toroidal dipole that involve the third-order differentiation [21]. It is hoped that our derivation will prove to be attractive due to its simplicity and for its obvious advantages in teaching applications as well as for research.

The starting point in our derivation is equations (1), (2). Equations similar to Jefimenko's were obtained by Panofsky and Phillips [18] and recently rederived by Heras [20]. Some attention to these formulae was given by McDonald [19], who also provided a comparison between them. The main difference (we do not discuss a derivation here) is the expression for the electric field

$$\mathbf{E}(\mathbf{r}, t) = \int \mathrm{d}\mathbf{r}' \left(\frac{[\rho]\hat{\mathbf{R}}}{R^2} + \frac{([\mathbf{j}] \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} + ([\mathbf{j}] \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{cR^2} + \frac{\left(\left[\frac{\partial \mathbf{j}}{\partial t'}\right] \times \hat{\mathbf{R}}\right) \times \hat{\mathbf{R}}}{c^2 R} \right). \tag{18}$$

This expression makes transparent the transversality of radiational fields (see equation (2)) and is definitely better to use in considering radiation problems. At first glance, it is also preferable for calculating the total electric field of a moving charge because the structure of the last term in (18), which looks similar to the structure of the 'acceleration' part of (14), and only in one term do we need to perform differentiation. However, the comparison of calculations with two expressions shows that Jefimenko's form is slightly simpler to work with. We are leaving this as a possible exercise which, in addition, gives as alternatives to equations (15), (16) expressions derived by the standard method in [22, 23].

Let us also comment on a lesser-known alternative expression for the fields of a point charge first discovered by Heaviside [24] and rediscovered by Feynman [25] (see also [12, 13]). They are

$$\mathbf{E}(\mathbf{r}, t) = e\left[\frac{\hat{\mathbf{R}}_s}{R_s^2}\right] + \frac{e[R_s]}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}_s}{R_s^2}\right] + \frac{e}{c^2}\frac{\partial^2}{\partial t^2}[\hat{\mathbf{R}}_s], \tag{19}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{e}{c}\left[\frac{\hat{\mathbf{R}}_s}{R_s}\right] \times \frac{\partial}{\partial t}[\hat{\mathbf{R}}_s] + \frac{e}{c^2}\hat{\mathbf{R}}_s \times \frac{\partial^2}{\partial t^2}[\hat{\mathbf{R}}_s]. \tag{20}$$

The peculiarity of equations (19), (20) is obvious. The fields are expressed entirely in terms of retarded position and their derivatives with respect to time t. These expressions, and their derivation and generalization were the subject of recent discussions in [5, 10, 11, 14].

We tried, without success, to find the transparent derivation of (19), (20) comparable to a given derivation of standard equations (15), (16). Looking for possible clues in Feynman's book, we found the following comments ([25], footnote on p 21).

> 'If you have a lot of paper and time you can try to work it through yourself. We would, then, make two suggestions: First, don't forget that the derivatives of r' are complicated, since it is a function of t'. Second, don't try to *derive* [original emphasis] (21.1) [our (19)], but carry out all of the derivatives in it, and then compare what you get with the **E** obtained from the potentials (21.33) and (21.34).'

The known derivations of Feynman's expression look exactly like Feynman's suggestion '...carry out all of the derivatives in it and compare...' but are presented in inverse order. The only difference is that the comparison is made not to (15) but to an intermediate result (9).

The main simplifying idea in our derivation is to hide all the derivatives inside 'retarded brackets', the Heaviside–Feynman form, in some sense, works in the opposite way: there is an increase in the number of derivatives outside brackets (in an attempt to express them only in terms of position) which is in some sense a step back and makes calculations even more complicated.

The presence of three terms in (19), which are, according to [12], especially useful because each term may be given a physical interpretation, is not a special characteristic of the Heaviside–Feynman expression. The standard equation (15) can also be presented as a sum of three terms. Moreover, the generalization of Heaviside–Feynman form for the case of a moving dipole leads to a term with no satisfactory physical interpretation [11, 12]. In addition, it seems that the analysis of some standard and important special cases, for example, $\dot{\mathbf{V}} \perp \mathbf{V}$

and $\dot{\mathbf{V}} \| \mathbf{V}$, is easy to perform with the standard expressions, as well as the calculation of power and angular distribution of radiation from a moving point particle, etc. These expressions, thus, are of a limited interest for teaching and for calculations, and we probably have to follow Feynman's advice: *'don't try to derive'!*

3. Conclusion

We have presented a straightforward and economical treatment of the derivation of the electric and magnetic fields of a moving point charge without recourse to implicit differentiation of retarded quantities, by making use of the general solution of Maxwell's equations in Jefimenko's form and a wise choice of the order of operations.

The derivation does not involve knowledge of methods that go beyond those usually at the disposal of undergraduate students, provides an illustration of some properties of the δ-function known to students from a first course in quantum mechanics, and gives an excellent example of when it is sometimes better to use the solution in the most general form and not to always attempt direct calculations. This result can be presented in one lecture in a very transparent way, in contrast to direct calculation with potentials.

Furthermore, it is felt that this approach can compete with the relativistic one in its clarity and computational advantages, and can make substantial simplifications in computationally more advanced problems [10, 11, 20, 21, 26].

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